Non-differentiable Degrees of Freedom : Fluctuating Metric Signature

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In this article we investigate the metric signature as a non-differentiable (*i.e.* discrete as opposed to continuous) degree of freedom. The specific model is a vacuum 7D Universe on the principal bundle with an SU(2) structural group. An analytical solution is found which to a 4D observer appears as a flat Universe with a fluctuating metric signature, and frozen extra dimensions with an SU(2) instanton gauge field. A piece of this solution with linear size of the Planck length ($\approx l_{Pl}$) can be considered as seeding the quantum birth of a regular Universe. A boundary of this piece can initiate the formation of a Lorentzian Universe filled with the gauge fields and in which the extra dimensions have been "frozen".

I. INTRODUCTION

Ref. [1] presented a model for the quantum birth of a 5D Universe from "Nothing" via a metric with fluctuating signature. In this scenario the 5^{th} dimension is associated with a U(1) gauge group (*i.e.* the 5D spacetime is the total space of the principal bundle with U(1) as the structural group). In this case the electromagnetic gauge field appeared as the non-diagonal components $G_{5\mu}$ ($\mu = 0, 1, 2, 3$) of the 5D metric. The basic idea in this model was that the signature of the metric, $\eta_{\bar{A}\bar{B}}$, ($\bar{A}, \bar{B} = 0, 1, 2, 3, 5, 6, \cdots$ are viel-bein indices) was an independent degree of freedom from the viel-bein

$$ds^2_{(MD)} = \eta_{\bar{A}\bar{B}} \left(h^{\bar{A}}_C dx^C \right) \left(h^{\bar{B}}_D dx^D \right) \tag{1}$$

C, D are the multidimensional (MD) coordinate indices, x^A are the coordinates on the total space of the principal bundle with a structural group $\mathcal{G}$. The metric can be rewritten

$$ds^2_{(MD)} = \eta_{\bar{a}\bar{b}} \left(h^{\bar{a}}_c dx^c + h^{\bar{a}}_\mu dx^\mu \right) \left(h^{\bar{b}}_c dx^c + h^{\bar{b}}_\mu dx^\mu \right) + \eta_{\bar{\mu}\bar{\nu}} \left(h^{\bar{\mu}}_\alpha dx^\alpha \right) \left(h^{\bar{\nu}}_\beta dx^\beta \right) \tag{2}$$

$\bar{a}, \bar{b}$ are the viel-bein indices for the fibre of the principal bundle, and c, d are the coordinate indices on the fibre; $\bar{\mu}, \bar{\nu}$ and α, β play the same role for the 4D base of the principal bundle. From Eqs. (1), (2) we see that $\eta_{\bar{A}\bar{B}}$ and $h^{\bar{A}}_B$ are the independent degrees of freedom. Also $h^{\bar{A}}_B$ is a continuous (differentiable) variable while $\eta_{\bar{A}\bar{B}}$ is a discrete (non-differentiable) variable. Thus the dynamics of the metric signature, $\eta_{\bar{A}\bar{B}}$, can not be described by differential equations; one should apply a quantum description for these degrees of freedom. This description could be stochastic in agreement with 't Hooft's proposition that the origin of quantum gravity should be stochastic [2].

In this case the basic question is: what kind of weight function should be associated with each mode ($\eta_{\bar{A}\bar{B}} = \pm 1$ in our case). We will assume [1], [3] that this weight is connected with the algorithmic complexity (AC) of a given mode. The notion of AC was first introduced by Kolmogorov [4] and leads to an algorithmic understanding of probability. The idea is simple: the probability for an object is connected with the *minimal* length of an algorithm describing this object. Kolmogorov showed how this definition could be used to define a notion of probability. Such a definition of probability **can be applied to a single object** and as such is of great interest for quantum gravity.

In this paper we expand the 5D model of [1] to 7D with an SU(2) gauge group. We consider a 7D Universe with a fluctuating metric signature ($\eta_{\bar{0}\bar{0}} = \pm 1$) and show that from the 4D point of view we obtain an SU(2) instanton field configuration and frozen extra dimensions (ED). The SU(2) solution given here in the context of higher dimensional gravity is related to the SU(2) [5] [6] and higher gauge groups [7] wormhole instantons solutions to the coupled Einstein-Yang-Mills (EYM) systems in 4D.

Refs. [5] [7] examined the cosmological consequences of these 4D EYM wormhole solutions. For example, Hosoya and Ogura [5] related their solution to a wormhole tunneling amplitude and the Coleman mechanism [8] for the vanishing of the cosmological constant. In this paper we will also investigate the cosmological consequences of our solution. We will argue that a small piece of our solution, with the linear size $\approx l_{Pl}$, can be interpreted as giving rise to the quantum birth of a Universe as a result of the fluctuating metric signature. Then the evolution of an ordinary Lorentzian Universe can begin from a boundary of this $\approx l_{Pl}$ sized piece. Simultaneously with the formation of this Lorentzian Universe the ED *split off*,

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i.e. the $h^{\bar{a}}_{b}$ components become non-dynamical variables. Once the ED become "frozen" the Lagrangian effectively reduces to ordinary 4D Einstein-Yang-Mills gravity. This scenario can be seen as the non-singular, quantum birth of a Universe from "Nothing" which results from fluctuations of the metric signature at the Planck scale.

II. FIELD EQUATIONS

The total space E of the principal bundle with structural group $\mathcal{G}$ can be taken as the space E which is acted on by $\mathcal{G}$. This group action determines the factor-space $\mathcal{H} = E/\mathcal{G}$ (the base of the principal bundle) with the 4D metric

$$ds^2_{(4)} = \eta_{\bar{\mu}\bar{\nu}} \left(h^{\bar{\mu}}_{\alpha} dx^{\alpha}\right) \left(h^{\bar{\nu}}_{\beta} dx^{\beta}\right) \tag{3}$$

which is the last term in Eq. (2). This allows us to insert a 4D term in the MD action

$$\begin{aligned} S = & \int (R + 2\Lambda_1) \sqrt{|G|} d^{4+N} x + \int (2\Lambda_2) \sqrt{|g|} d^4 x \\ = & \int \left[\int (R + 2\Lambda_1) \sqrt{|\gamma|} d^N y + 2\Lambda_2 \right] \sqrt{|g|} d^4 x \end{aligned} \tag{4}$$

R is the Ricci scalar and $G_{AB} = \eta_{\bar{C}\bar{D}} h^{\bar{C}}_{A} h^{\bar{D}}_{B}$ is the MD metric on the total space; $g_{\mu\nu} = \eta_{\bar{\alpha}\bar{\beta}} h^{\bar{\alpha}}_{\mu} h^{\bar{\beta}}_{\nu}$ is the 4D metric on the base of the principal bundle; $\gamma_{ab} = \eta_{\bar{c}\bar{d}} h^{\bar{c}}_{a} h^{\bar{d}}_{b}$ is the metric on $\mathcal{G}$; G, g and γ are the appropriate metric determinates; $\Lambda_{1,2}$ are the MD and 4D λ-constants; $N = dim(\mathcal{G})$. The MD action of Eq. (4) has several points in common with the 4D EYM action considered in Ref. [5] (non-zero cosmological constants and effective SU(2) "Yang-Mills" gauge fields). Eq. (4) also has a connection to the action for the Non-gravitating Vacuum Energy Theory [9]. In Ref. [9] Guendelman considers an action which has degrees of freedom which are independent of the metric, with the resulting action having two measures of integration (involving metric and non-metric degrees of freedom). Eq. (4) incorporates two distinct degrees of freedom : the continuous variables, $h^{\bar{A}}_{B}$, and the discrete variables, $\eta_{\bar{A}\bar{B}}$. In Ref. [9] both the metric and non-metric degrees of freedom were continuous.

This choice of the action indicates that we restrict our coordinate transformation law to

$$y'^a = y'^a \left(y^b\right) + f^a \left(x^{\alpha}\right), \tag{5}$$

$$x'^{\mu} = x'^{\mu} \left(x^{\alpha}\right). \tag{6}$$

These coordinate transformations do not destroy the $\mathcal{G}$-structure of the total space of the principal bundle, *i.e.* they do not mix different fibres of the bundle.

The independent, *continuous degrees* of freedom are: the vier-bein $h^{\bar{\mu}}_{\nu}(x^{\alpha})$, the gauge potential $h^{\bar{a}}_{\mu}(x^{\alpha})$ and the scalar field $b(x^{\alpha})$ which is defined as

$$h^{\bar{a}}_{b}(x^{\mu}) = \sqrt{b(x^{\mu})} e^{\bar{a}}_{b} \tag{7}$$

$e^{\bar{a}}_{b}$ is defined as

$$\omega^{\bar{a}} = e^{\bar{a}}_{b} dx^b \tag{8}$$

x^b are the coordinates on the group $\mathcal{G}$; $\omega^{\bar{a}}$ are the 1-forms satisfying

$$d\omega^{\bar{a}} = f^{\bar{a}}_{\bar{b}\bar{c}} \omega^{\bar{b}} \wedge \omega^{\bar{c}} \tag{9}$$

$f^{\bar{a}}_{\bar{b}\bar{c}}$ are the structural constants of $\mathcal{G}$. Varying the action in Eq. (4) with respect to $h^{\bar{\mu}}_{\nu}$, $h^{\bar{a}}_{\nu}$ and b leads to (see the Appendix for details)

$$R_{\bar{\mu}\bar{\nu}} - \frac{1}{2}\eta_{\bar{\mu}\bar{\nu}} R = \eta_{\bar{\mu}\bar{\nu}} \left(\Lambda_1 + \frac{\Lambda_2}{b^{3/2}}\right), \tag{10}$$

$$R_{\bar{a}\bar{\mu}} = 0, \tag{11}$$

$$R^{\bar{a}}_{\bar{a}} = -\frac{6}{5}\left(\Lambda_1 + \frac{\Lambda_2}{b^{3/2}}\right). \tag{12}$$

Eq. (10) are the Einstein vacuum equations with λ-terms; Eq. (11) are the "Yang-Mills" equations; Eq. (12) is reminiscent of Brans-Dicke theory since the metric on each fibre is symmetric and has only one degree of freedom - the scalar factor $b(x^{\mu})$ defined in Eq. (7).

We now investigate Eqs. (10)-(12) using the ansatz

$$ds^2 = \sigma dt^2 + b(t)\left(\omega^{\bar{a}} + A^{\bar{a}}_{\mu} dx^{\mu}\right)\left(\omega_{\bar{a}} + A_{\bar{a}\mu} dx^{\mu}\right) + a(t) d\Omega^2_3 \tag{13}$$

$\sigma = \pm 1$ describes the possible quantum fluctuation of the metric signature between Euclidean and Lorentzian modes, $A^{\bar{a}}_{\mu}$ are SU(2) gauge potentials, $d\Omega_3^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is the metric on the unit S^3 sphere and $x^0 = t, x^1 = \chi, x^2 = \theta, x^3 = \phi, x^5 = \alpha, x^6 = \beta, x^7 = \gamma$. ($\alpha, \beta, \gamma$ are the Euler angles for the SU(2) group)

$$\omega^1 = \frac{1}{2}(\sin\alpha d\beta - \sin\beta\cos\alpha d\gamma), \tag{14}$$

$$\omega^2 = -\frac{1}{2}(\cos\alpha d\beta + \sin\beta\sin\alpha d\gamma), \tag{15}$$

$$\omega^3 = \frac{1}{2}(d\alpha + \cos\beta d\gamma). \tag{16}$$

The nondiagonal components of the MD metric take the instanton-like form [6] [10] :

$$A^a_\chi = \frac{1}{4}\left\{-\sin\theta\cos\varphi; -\sin\theta\sin\varphi; \cos\theta\right\}(f(t)-1), \tag{17}$$

$$A^a_\theta = \frac{1}{4}\left\{-\sin\varphi; -\cos\varphi; 0\right\}(f(t)-1), \tag{18}$$

$$A^a_\varphi = \frac{1}{4}\left\{0; 0; 1\right\}(f(t)-1). \tag{19}$$

Substituting into Eqs. (10)-(12) gives

$$\frac{1}{3}R^{\bar{a}}_{\bar{a}} = R^{\bar{5}}_{\bar{5}} = -\frac{\sigma}{2}\frac{\ddot{b}}{b} + \frac{2}{b} - \frac{\sigma}{4}\frac{\dot{b}^2}{b^2} - \frac{3}{4}\sigma\frac{\dot{a}\dot{b}}{ab} + \frac{1}{8}\frac{b}{a}\left(\sigma E^2 + H^2\right) = -\frac{2}{5}\left(\Lambda_1 + \frac{2\Lambda_2}{b^{3/2}}\right), \tag{20}$$

$$G_{\bar{0}\bar{0}} = -3\frac{\sigma}{b} + \frac{3}{4}\frac{\dot{b}^2}{b^2} - 3\frac{\sigma}{a} + \frac{9}{4}\frac{\dot{a}\dot{b}}{ab} + \frac{3}{16}\frac{\dot{a}^2}{a^2} - \frac{3}{16}\frac{b}{a}\left(E^2 - \sigma H^2\right) = \sigma\left(\Lambda_1 + \frac{\Lambda_2}{b^{3/2}}\right), \tag{21}$$

$$G_{\bar{1}\bar{1}} = \frac{3}{2}\sigma\frac{\ddot{b}}{b} - \frac{3}{b} + \sigma\frac{\ddot{a}}{a} - \frac{1}{a} + \frac{3}{2}\sigma\frac{\dot{a}\dot{b}}{ab} - \frac{\sigma}{4}\frac{\dot{a}^2}{a^2} + \frac{1}{16}\frac{b}{a}\left(\sigma E^2 - H^2\right) = \left(\Lambda_1 + \frac{\Lambda_2}{b^{3/2}}\right), \tag{22}$$

$$G_{\bar{2}\bar{7}} = 2\ddot{f} + 5\frac{\dot{b}\dot{f}}{b} + \frac{\dot{a}\dot{f}}{a} - 4\frac{\sigma}{a}f\left(f^2 - 1\right) = 0, \tag{23}$$

$$E^2 = E^a_i E^{ai} = \dot{f}^2, \quad H^2 = H^a_i H^{ai} = \frac{\left(f^2-1\right)^2}{a}, \tag{24}$$

$G_{\bar{A}\bar{B}} = R_{\bar{A}\bar{B}} - (1/2)\eta_{\bar{A}\bar{B}}R$; $i = 1, 2, 3$ are space indices; the "electromagnetic" fields are

$$E^a_i = F^a_{0i}, \quad H^a_i = \frac{1}{2}\varepsilon_{ijk}F^{ajk} \tag{25}$$

$F^a_{\mu\nu}$ is the field strength tensor for the non-Abelian gauge group. The wormhole instanton of Ref. [5] had a vanishing "electric" field. In contrast the solution studied here has both non-vanishing "electric" and "magnetic" fields.

III. DEFINITION OF ALGORITHMIC COMPLEXITY

We now examine the dynamic behavior of the *discrete* quantity σ which describes the quantum fluctuations (trembling) between Euclidean and Lorentzian modes. One fruitful approach is the stochastic approach proposed by 't Hooft [2]. The main question is: how to define a weight function for each mode (Euclidean and Lorentzian) ? Our proposition is that these weight functions be given by the AC of the Eqs. (20)-(23).

In detail Eqs. (20)-(23) define the dynamic behavior of the continuous variables $a(t), b(t)$ and $f(t)$. Each equation oscillates between the two possibilities $\sigma = \pm 1$, and when viewed as an algorithm, will have an AC which depends on the value of σ. Based on the AC, each equation is assigned two weight functions: one for $\sigma = +1$ and one for $\sigma = -1$. Certain equations will be simpler in the Euclidean mode while others will be simpler in the Lorentzian mode. A common example of this behavior is the Polyakov-'t Hooft instanton which exists only in Euclidean space.

Kolmogorov's [4] definition for AC is :

The algorithmic complexity $K(x \mid y)$ of the object x for a given object y is the minimal length of the "program" P that is written as a sequence of the zeros and unities which allows us to construct x having y:

$$K(x \mid y) = \min_{A(P,y)=x} l(P) \tag{26}$$

where $l(P)$ is length of the program P; $A(P, y)$ is the algorithm for calculating an object x, using the program P, when the object y is given.

This definition gives an exact mathematical meaning to the word "simple". It is also in the spirit of Einstein's statement: *"Everything should be simple as possible but not more"*.

IV. QUANTUM FLUCTUATION FOR THE INITIAL EQUATIONS

Our assumption of quantum trembling between Euclidean and Lorentzian modes is described by as a quantum-stochastic fluctuation between the equations

$$
\begin{array}{ccc}
\sigma = +1 & \longleftrightarrow & \sigma = -1 \\
 & \Downarrow & \\
(R^{+})^{\bar{5}}_{\bar{5}} & \longleftrightarrow & (R^{-})^{\bar{5}}_{\bar{5}} \\
(G^{+})_{\bar{0}\bar{0}} & \longleftrightarrow & (G^{-})_{\bar{0}\bar{0}} \\
(G^{+})_{\bar{1}\bar{1}} & \longleftrightarrow & (G^{-})_{\bar{1}\bar{1}} \\
(G^{+})_{\bar{2}\bar{7}} & \longleftrightarrow & (G^{-})_{\bar{2}\bar{7}}
\end{array}
\tag{27}
$$

The signs ($\pm$) denote the equations of the Euclidean (+) or Lorentzian (-) mode. Now we define the weight functions for each pair in Eqs. (27).

A. $G_{\bar{2}\bar{7}}$ equation

This equation in the Euclidean mode is

$$2\ddot{f} + 5\frac{\dot{b}\dot{f}}{b} + \frac{\dot{a}\dot{f}}{a} - \frac{4}{a} f\left(f^2 - 1\right) = 0 \tag{28}$$

which has the instanton solution

$$\dot{f} = \frac{1 - f^2}{\sqrt{a}}, \tag{29}$$

where

$$b = b_0 = const \tag{30}$$

Eq. (29) implies the instanton condition

$$E^a_i E^i_a = H^a_i H^i_a. \tag{31}$$

In the Lorentzian mode

$$2\ddot{f} + 5\frac{\dot{b}\dot{f}}{b} + \frac{\dot{a}\dot{f}}{a} + \frac{4}{a} f\left(f^2 - 1\right) = 0 \tag{32}$$

and the instanton solution (31) is not a solution of (32). It is well known that the non-singular, instanton solution exists only in Euclidean space.

In terms of the AC criteria the Euclidean equation (28) is simpler than Lorentzian equation (32), since it is equivalent to the first order differential equation (29).

In a first rough approximation we set the probability of the $G_{\bar{2}\bar{7}} = 0$ equation for the Euclidean mode to $p^{+}_{27} = 1$ and the Lorentzian mode to $p^{-}_{27} = 0$.

The exact definition for each $p^{\pm}_{AB}$ probability is [3]

$$p^{\pm}_{AB} = \frac{e^{-K^{\pm}_{AB}}}{e^{-K^{+}_{AB}} + e^{-K^{-}_{AB}}} \tag{33}$$

where $K^{\pm}_{AB}$ is the AC for the appropriate equation. If $K^{+}_{27} \ll K^{-}_{27}$ we have $p^{+}_{27} = 1$ and $p^{-}_{27} = 0$. The expression for the probability in Eq. (33) can be seen as the discrete variable analog of the Euclidean path integral transition probability. For a continuous variable the Euclidean path integral gives the probability for the variable to evolve from some initial configuration to some final configuration as being proportional to the exponential of minus the action ($\propto e^{-S}$). Eq. (33) is similar, but with the AC replacing the action. The denominator normalizes the probability (it is a sum rather than integral since we are dealing with a discrete variable).

B. $R^{\bar{5}}_{\bar{5}}$ equation

This equation in the Euclidean mode is

$$-\frac{1}{2}\frac{\ddot{b}}{b} + \frac{2}{b} - \frac{1}{4}\frac{\dot{b}^2}{b^2} - \frac{3}{4}\frac{\dot{a}\dot{b}}{ab} + \frac{1}{8}\frac{b}{a}\left(E^2 + H^2\right) = -\frac{2}{5}\left(\Lambda_1 + \frac{2\Lambda_2}{b^{3/2}}\right), \tag{34}$$

and in the Lorentzian mode

$$\frac{1}{2}\frac{\ddot{b}}{b}+\frac{2}{b}+\frac{1}{4}\frac{\dot{b}^2}{b^2}+\frac{3}{4}\frac{\dot{a}\dot{b}}{ab}+\frac{1}{8}\frac{b}{a}\left(-E^2+H^2\right)=-\frac{2}{5}\left(\Lambda_1+\frac{2\Lambda_2}{b^{3/2}}\right), \tag{35}$$

The Lorentzian mode equation is simpler because the two last terms annihilate as a consequence of the instanton condition (31).

To a first rough approximation we set the probability of the $R^{\bar{5}}_{\bar{5}}$ equation for the Euclidean mode to $p^+_{55}=0$ and the Lorentzian mode to $p^-_{55}=1$.

C. $G_{\bar{0}\bar{0}}$ equation

This equation in the Euclidean mode is

$$-\frac{3}{b}+\frac{3}{4}\frac{\dot{b}^2}{b^2}-\frac{3}{a}+\frac{9}{4}\frac{\dot{a}\dot{b}}{ab}+\frac{3}{16}\frac{\dot{a}^2}{a^2}-\frac{3}{16}\frac{b}{a}\left(E^2-H^2\right)=\left(\Lambda_1+\frac{\Lambda_2}{b^{3/2}}\right) \tag{36}$$

and in the Lorentzian mode

$$\frac{3}{b}+\frac{3}{4}\frac{\dot{b}^2}{b^2}+\frac{3}{a}+\frac{9}{4}\frac{\dot{a}\dot{b}}{ab}+\frac{3}{16}\frac{\dot{a}^2}{a^2}-\frac{3}{16}\frac{b}{a}\left(E^2+H^2\right)=-\left(\Lambda_1+\frac{\Lambda_2}{b^{3/2}}\right). \tag{37}$$

In this case because of the instanton condition (31) the Euclidean equation is simpler and therefore in the first rough approximation we can set the probability of the $G_{\bar{0}\bar{0}}=0$ equation for the Euclidean mode to $p^+_{00}=1$ and the Lorentzian mode to $p^-_{00}=0$.

D. $G_{\bar{1}\bar{1}}$ equation

This equation in the Euclidean mode is

$$\frac{3}{2}\frac{\ddot{b}}{b}-\frac{3}{b}+\frac{\ddot{a}}{a}-\frac{1}{a}+\frac{3}{2}\frac{\dot{a}\dot{b}}{ab}-\frac{1}{4}\frac{\dot{a}^2}{a^2}+\frac{1}{16}\frac{b}{a}\left(E^2-H^2\right)=\left(\Lambda_1+\frac{\Lambda_2}{b^{3/2}}\right) \tag{38}$$

and in the Lorentzian mode

$$-\frac{3}{2}\frac{\ddot{b}}{b}-\frac{3}{b}-\frac{\ddot{a}}{a}-\frac{1}{a}-\frac{3}{2}\frac{\dot{a}\dot{b}}{ab}+\frac{1}{4}\frac{\dot{a}^2}{a^2}-\frac{1}{16}\frac{b}{a}\left(E^2+H^2\right)=\left(\Lambda_1+\frac{\Lambda_2}{b^{3/2}}\right). \tag{39}$$

As in the previous subsection as a consequence of the instanton condition (31) the Euclidean mode is simpler. Therefore in the first rough approximation we set $p^+_{11}=1$ and $p^-_{11}=0$.

E. Mixed system of equations

The mixed system of equations for the 7D spacetime with fluctuating metric signature is

$$2\ddot{f}+5\frac{\dot{b}\dot{f}}{b}+\frac{\dot{a}\dot{f}}{a}-\frac{4}{a}f\left(f^2-1\right)=0, \tag{40}$$

$$\frac{1}{2}\frac{\ddot{b}}{b}+\frac{2}{b}+\frac{1}{4}\frac{\dot{b}^2}{b^2}+\frac{3}{4}\frac{\dot{a}\dot{b}}{ab}+\frac{1}{8}\frac{b}{a}\left(-E^2+H^2\right)=-\frac{2}{5}\left(\Lambda_1+\frac{2\Lambda_2}{b^{3/2}}\right), \tag{41}$$

$$-\frac{3}{b}+\frac{3}{4}\frac{\dot{b}^2}{b^2}-\frac{3}{a}+\frac{9}{4}\frac{\dot{a}\dot{b}}{ab}+\frac{3}{16}\frac{\dot{a}^2}{a^2}-\frac{3}{16}\frac{b}{a}\left(E^2-H^2\right)=\left(\Lambda_1+\frac{\Lambda_2}{b^{3/2}}\right), \tag{42}$$

$$\frac{3}{2}\frac{\ddot{b}}{b}-\frac{3}{b}+\frac{\ddot{a}}{a}-\frac{1}{a}+\frac{3}{2}\frac{\dot{a}\dot{b}}{ab}-\frac{1}{4}\frac{\dot{a}^2}{a^2}+\frac{1}{16}\frac{b}{a}\left(E^2-H^2\right)=\left(\Lambda_1+\frac{\Lambda_2}{b^{3/2}}\right). \tag{43}$$

The solution for this system is

$$a=t^2, \tag{44}$$

$$f=\frac{t^2-t_0^2}{t^2+t_0^2}, \tag{45}$$

$$b=b_0=const, \tag{46}$$

$$\Lambda_1=-\frac{1}{b_0}, \tag{47}$$

$$\Lambda_2=-2\sqrt{b_0}. \tag{48}$$

The existence of this solution is somewhat surprising ! Let us clarify this. Normally in any dimension the Bianchi identities are fulfilled. Therefore some gravitational field equations are not independent of the others. Ordinarily the superfluous equations are associated with initial conditions (*i.e.* Eq. (42) above). In our case the mixed system above comes from a model with a varying metric signature. As a consequence the Bianchi identities are not correct and this system should be unsolvable. Evidently the solution is a condition for the solvability of the mixed system which uniquely define the Λ-constants. If the solution in Eqs. (44)-(48) is unique then it must be absolutely stable.

The physical meaning of this solution is:

- Eq. (44) implies a flat 4D Einstein spacetime that is not effected by matter.
- Eq. (45) implies a Polyakov - 't Hooft instanton gauge field configuration which is not effected by gravity.
- Eq. (46) implies a frozen ED.
- Eqs. (47)-(48) imply that the dynamical equations uniquely determine the $\Lambda_{1,2}$-constants.

It is interesting to note that the effective cosmological constant terms on the right hand side of Eqs. (10) (12) (*i.e.* Λ_1 and $\Lambda_2/b^{3/2}$) are inversely proportional to the size of the ED, b_0. Thus in order to have a small cosmological constant term one needs to have a large ED. This could be seen as supporting the large extra dimensions scenarios [11].

V. PHYSICAL APPLICATIONS OF THE SOLUTION

A. Regular Universe

We can interpret this solution as a flat 4D Universe with fluctuating metric signature, filled with an SU(2) instanton gauge field and frozen ED. Astonishingly this Universe has only one manifestation of gravity: the frozen ED that result from the fluctuating metric signature. This model Universe is a simple example of possible effects connected with the dynamics of non-differentiable variables.

B. Non-singular birth of the Universe

Various researchers (see Ref. [12] for example) have speculated about the quantum birth of the Universe from "Nothing". In light of this we can interpret a small piece (with linear size $\approx l_{Pl}$) of our model 7D Universe as a quantum birth of the regular 4D Universe. In contrast to other scenarios this origin has a metric signature trembling between Euclidean and Lorentzian modes. Further we postulate that on a boundary of this origin there occurs

- *a quantum transition* to only one Lorentzian mode of fixed metric signature.
- *a splitting off* the ED so that the metric on the fibres ($h^{\bar{a}}_{\bar{b}}$) becomes a non-dynamical variable. After this splitting off the linear size of the gauge group remains constant yielding ordinary 4D Einstein-Yang-Mills gravity.

These assumptions about a quantum transition from fluctuating metric signature $(\pm 1, +1, \cdots, +1)$ to Lorentzian signature $(-1, +1, \cdots, +1)$ and a splitting off of the ED should not be seen as something extraordinary and new, but rather as an extension of our postulate about the quantum birth of the regular 4D Universe, discussed above, with certain laws (gravitational equations + non-differentiable dynamic). The present case can be seen a quantum-stochastic change or evolution of these laws (here this involves only the quantum transition of η_{00} and the splitting off of the ED).

The probability for the quantum birth is

$$P \approx N e^{-S} \tag{49}$$

where S is the Euclideanized, dimensionless action, which should be $S \approx 1$ in Planck units. The prefactor N is of more interest, since it contains information about the topological structure of the boundary of the origin.

The probability for the quantum-stochastic transition to Lorentzian mode and splitting off of the ED should be determined by the AC of the final and initial states. Such a quantum-stochastic transition can occur only if the final state with Lorentzian mode and splitting off the ED is simpler than the initial state with the fluctuating metric signature and dynamic ED.

VI. CONCLUSIONS

In this article we have investigated possible quantum gravity effects connected with *non-differentiable degrees of freedom* [1], [3]. By considered the quantum trembling of the metric signature of a 7D model Universe we have found a solution which describes *a flat 4D Universe with a fluctuating metric signature filled with an SU(2) instanton gauge field and frozen ED.* A piece of this solution can be considered *as resulting in the quantum birth of the regular Universe* with the fluctuating metric signature. An important peculiarity for this model is that it is a *vacuum* model without any kind of matter; only the gauge field appear as non-diagonal components of the MD metric. This is in the spirit of Einstein's point of view that Nature consists of "Nothing".

VII. APPENDIX

We start from the Lagrangian adopted for the vacuum gravitational theory on the principal bundle with the structural group $\mathcal{G}$ ($\dim(\mathcal{G}) = N$). $\mathcal{G}$ is the gauge group associated with the EDs

$$S = \int (R + 2\Lambda_1)\sqrt{|G|}d^{4+N}x + \int (2\Lambda_2')\sqrt{|g|}d^4x \tag{50}$$

where R is the Ricci scalar for the total space; G and g are the determinant of the metric on the total space and base of the principal bundle respectively, Λ_1, Λ_2' are the MD and 4D λ-constants. This Lagrangian is correct if the coordinate transformations conserve the topological structure of the total space (*i.e.* does not mix the fibres)

$$y'^a = y'^a\left(y^b\right) + f^a\left(x^\alpha\right), \tag{51}$$

$$x'^\mu = x'^\mu\left(x^\alpha\right). \tag{52}$$

The metric on the total space can be written as

$$ds^2_{(MD)} = \left(\sqrt{b}\omega^{\bar{a}} + h^{\bar{a}}_\mu dx^\mu\right)\left(\sqrt{b}\omega_{\bar{a}} + h_{\bar{a}\mu}dx^\mu\right) + \left(h^{\bar{\mu}}_\alpha dx^\alpha\right)\left(h_{\bar{\mu}\beta}dx^\beta\right) \tag{53}$$

$$\omega^{\bar{a}} = e^{\bar{a}}_b dx^b \quad h^{\bar{a}}_b = \sqrt{b(x^\mu)}e^{\bar{a}}_b \tag{54}$$

where x^μ and y^b are the coordinates along the base and fibres respectively; (Greek indices)$= 0, 1, 2, 3$ and (Latin indices)$=$ $5, 6, \cdots, N$; $\bar{A} = \bar{a}, \bar{\mu}$ is the viel-bein index; $\eta_{\bar{A}\bar{B}} = \{\pm1, \pm1, \cdots, \pm1\}$ is the signature of the MD metric; $\omega^{\bar{a}}$ are the 1-forms satisfying to the structural equations

$$d\omega^{\bar{a}} = f^{\bar{a}}_{\bar{b}\bar{c}}\omega^{\bar{b}} \wedge \omega^{\bar{c}} \tag{55}$$

where $f^{\bar{a}}_{\bar{b}\bar{c}}$ are the structural constants for the gauge group $\mathcal{G}$.

The independent degrees of freedom for gravity on the principal bundle with the structural group $\mathcal{G}$ is vier-bein $h^{\bar{\mu}}_\nu(x^\alpha)$, gauge potential $h^{\bar{a}}_\nu(x^\alpha)$ and scalar field $b(x^\alpha)$ [13–15]. All functions depend only on the point x^μ on the base of the principal bundle as a consequence of the symmetry of the fibres.

Varying the action (50) with respect to $h^{\bar{\mu}}_\nu(x^\alpha)$ leads to

$$\int \left(R^\mu_{\bar{\nu}} - \frac{1}{2}h^\mu_{\bar{\nu}}R - \Lambda_1 h^\mu_{\bar{\nu}}\right)\sqrt{|\gamma|}d^N y - \Lambda_2' h^\mu_{\bar{\nu}} = 0 \tag{56}$$

where $|\gamma| = \det h^{\bar{a}}_b = b^N \det e^{\bar{a}}_b$ is the volume element on the fibre and $\sqrt{|G|} = \sqrt{|g|}\sqrt{|\gamma|}$ is a consequence of the following structure of the MD metric

$$h = h^{\bar{A}}_B = \begin{pmatrix} h^{\bar{a}}_b & h^{\bar{a}}_\mu \\ 0 & h^{\bar{\nu}}_\mu \end{pmatrix}, \tag{57}$$

$$h^{-1} = h^B_{\bar{A}} = \begin{pmatrix} h^b_{\bar{a}} & -h^b_{\bar{a}}h^{\bar{a}}_\nu h^\nu_{\bar{\nu}} \\ 0 & h^\mu_{\bar{\nu}} \end{pmatrix}, \tag{58}$$

$$h^b_{\bar{a}} = \left(h^{\bar{a}}_b\right)^{-1} \quad h^\mu_{\bar{\nu}} = \left(h^{\bar{\nu}}_\mu\right)^{-1}. \tag{59}$$

An integration over the EDs can be easily performed since no functions depend on y^a

$$\int (\cdots)\sqrt{|\gamma|}d^N y = (\cdots)\int \sqrt{|\gamma|}d^N y = (\cdots)\, b^{N/2}V_{\mathcal{G}} \tag{60}$$

where $V_{\mathcal{G}} = \int \sqrt{\det(e^{\bar{a}}_b)}d^N y$ is the volume of the gauge group $\mathcal{G}$. In this case Eq. (56) becomes

$$R^\mu_{\bar{\nu}} - \frac{1}{2}h^\mu_{\bar{\nu}}R = \left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right)h^\mu_{\bar{\nu}} \tag{61}$$

where $\Lambda_2' = V_{\mathcal{G}}\Lambda_2$.

Varying with respect to $h^{\bar{a}}_\mu(x^\alpha)$ leads to

$$R^\mu_{\bar{a}} = 0 \tag{62}$$

as $h^{\bar{a}}_\mu$ does not consists in $\det(h^{\bar{A}}_B) = \det(h^{\bar{a}}_b)\det(h^{\bar{\mu}}_\nu)$.

Varying with respect to $b(x^\alpha)$ leads to

$$\frac{\delta S}{\delta b} = \sum_{\bar{a},b}\frac{\delta h^{\bar{a}}_b}{\delta b}\frac{\delta S}{\delta h^{\bar{a}}_b} = h^{\bar{a}}_A\left(R^A_{\bar{a}} - \frac{1}{2}h^A_{\bar{a}} - \Lambda_1 h^A_{\bar{a}}\right) \tag{63}$$

here we used Eq. (61) and $h^{\mu}_{\bar a} =$. This equation we write in the form

$$R^{\bar a}_{\bar a} - \frac{N}{2} R = N\Lambda_1 \tag{64}$$

From Eq. (61) we have

$$h^{\bar\nu}_{\mu}\left[R^{\mu}_{\bar\nu} - \frac{1}{2}h^{\mu}_{\bar\nu}R - \left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right)h^{\mu}_{\bar\nu}R\right] = h^{\bar\nu}_{\mu}\left[\cdots\right] + h^{\bar\nu}_{a}\left[\cdots\right] =$$

$$h^{\bar\nu}_{A}\left[R^{A}_{\bar\nu} - \frac{1}{2}h^{A}_{\bar\nu}R - \left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right)h^{A}_{\bar\nu}R\right] = R^{\bar\nu}_{\bar\nu} - 2R - 4\left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right) = 0 \tag{65}$$

Adding Eqs. (65) and (64) we find

$$R = R^{\bar A}_{\bar A} = -\frac{2}{N+2}\left[(N+4)\Lambda_1 + \frac{4\Lambda_2}{b^{N/2}}\right] \tag{66}$$

Finally we have

$$R^{\bar a}_{\bar a} = -\frac{2N}{N+2}\left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right), \tag{67}$$

$$R^{\mu}_{\bar a} = 0 \tag{68}$$

$$R^{\mu}_{\bar\nu} - \frac{1}{2}h^{\mu}_{\bar\nu}R = \left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right)h^{\mu}_{\bar\nu} \tag{69}$$

This equation system can be rewritten as

$$R^{\bar a}_{\bar a} = -\frac{2N}{N+2}\left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right), \tag{70}$$

$$R_{\bar\mu\bar a} = 0 \tag{71}$$

$$R_{\bar\mu\bar\nu} - \frac{1}{2}\eta_{\bar\mu\bar\nu}R = \left(\Lambda_1 + \frac{\Lambda_2}{b^{N/2}}\right)\eta_{\bar\mu\bar\nu} \tag{72}$$

here we have used $h^{\bar\nu}_{b} = 0$.

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[1] V.Dzhunushaliev, "Nondifferentiable Dynamic: Two Examples", gr-qc/9912018 (to be published in Gen. Rel. Gravt.)
[2] G. 't Hooft, Class. Quant. Grav. **16**, 3263 (1999).
[3] V. Dzhunushaliev, Class. Quant. Grav. **15**, 603 (1998).
[4] A. Kolmogorov, *"Information Theory and Algorithm Theory: Logical Foundation of the Information Theory. Combinatorial Foundation of the Information Theory and the Probability Calculus"* (Nauka, Moscow, 1987).
[5] A. Hosoya and W. Ogura, Phys. Lett. **B225**, 117 (1989)
[6] E.E. Donets and D.V. Gal'tsov, Phys. Lett. **B294**, 44 (1992)
[7] O. Bertolami *et. al.*, Int. J. Mod. Phys. **A6**, 4149 (1991)
[8] S. Coleman, Nucl. Phys. **B310**, 867 (1988)
[9] E.I. Guendelman, Mod. Phys. Lett. **A14**, 1043 (1999); "The Volume Element of Space-Time and Scale Invariance", hep-th/0011049 (Contribution to IARD 2000 Conference)
[10] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Y.S. Tyupkin, Phys. Lett. **B59**, 85 (1975)
[11] N. Arkani-Hamed *et. al.*, Phys. Lett. **B480**, 193 (2000)
[12] R. Bousso and S. Hawking, Int. J. Theor. Phys., **38**, 1227 (1999)
[13] A. Salam and J. Strathdee, Ann. Phys. **141**, 316 (1982).
[14] R. Percacci, J. Math. Phys. **24**, 807 (1983).
[15] R. Coquereaux and A. Jadczyk, Commun. Math. Phys. **90**, 79 (1983).